Problems and Solutions, JBMO 2014

Problem 1. *Find all distinct prime numbers* p, q *and* r *such that*

$$3p^4 - 5q^4 - 4r^2 = 26.$$

Solution. First notice that if both primes q and r differ from 3, then $q^2 \equiv r^2 \equiv 1 (\bmod\ 3)$, hence the left hand side of the given equation is congruent to zero modulo 3, which is impossible since 26 is not divisible by 3. Thus, $q = 3$ or $r = 3$. We consider two cases.

Case 1. $q = 3$.

The equation reduces to $3p^4 - 4r^2 = 431 \qquad (1)$.

If $p \neq 5$, by Fermat's little theorem, $p^4 \equiv 1 \ (\bmod\ 5)$, which yields $3 - 4r^2 \equiv 1 \ (\bmod\ 5)$, or equivalently, $r^2 + 2 \equiv 0 \ (\bmod\ 5)$. The last congruence is impossible in view of the fact that a residue of a square of a positive integer belongs to the set $\{0, 1, 4\}$. Therefore $p = 5$ and $r = 19$.

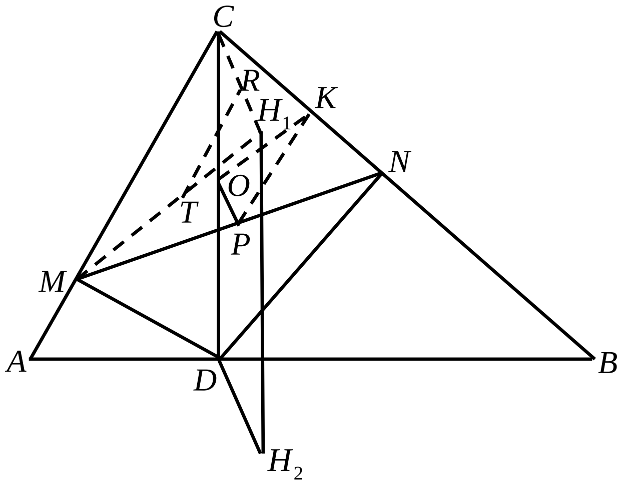
Case 2. $r = 3$.

The equation becomes $3p^4 - 5q^4 = 62 \qquad (2)$.

Obviously $p \neq 5$. Hence, Fermat's little theorem gives $p^4 \equiv 1 \ (\bmod\ 5)$. But then $5q^4 \equiv 1 \ (\bmod\ 5)$, which is impossible.

Hence, the only solution of the given equation is $p = 5$, $q = 3$, $r = 19$.

Problem 2. *Consider an acute triangle ABC with area S. Let $CD \perp AB$ $(D \in AB)$, $DM \perp AC$ $(M \in AC)$ and $DN \perp BC$ $(N \in BC)$. Denote by H_1 and H_2 the orthocentres of the triangles MNC and MND respectively. Find the area of the quadrilateral AH_1BH_2 in terms of S.*



Solution 1. Let O, P, K, R and T be the mid-points of the segments CD, MN, CN, CH_1 and MH_1, respectively. From ΔMNC we have that $\overline{PK} = \frac{1}{2}\overline{MC}$ and $PK \parallel MC$. Analogously, from ΔMH_1C we have that $\overline{TR} = \frac{1}{2}\overline{MC}$ and $TR \parallel MC$. Consequently, $\overline{PK} = \overline{TR}$ and $PK \parallel TR$. Also $OK \parallel DN$ (from ΔCDN) and since $DN \perp BC$ and $MH_1 \perp BC$, it follows that $TH_1 \parallel OK$. Since O is the circumcenter of ΔCMN, $OP \perp MN$. Thus, $CH_1 \perp MN$ implies $OP \parallel CH_1$. We conclude $\Delta TRH_1 \cong \Delta KPO$ (they have parallel sides and $\overline{TR} = \overline{PK}$), hence $\overline{RH_1} = \overline{PO}$, i.e. $\overline{CH_1} = 2\overline{PO}$ and $CH_1 \parallel PO$.

Analogously, $\overline{DH_2} = 2\overline{PO}$ and $DH_2 \parallel PO$. From $\overline{CH_1} = 2\overline{PO} = \overline{DH_2}$ and $CH_1 \parallel PO \parallel DH_2$ the quadrilateral CH_1H_2D is a parallelogram, thus $\overline{H_1H_2} = \overline{CD}$ and $H_1H_2 \parallel CD$. Therefore the area of the quadrilateral AH_1BH_2 is $\frac{\overline{AB} \cdot \overline{H_1H_2}}{2} = \frac{\overline{AB} \cdot \overline{CD}}{2} = S$.

Solution 2. Since $MH_1 \parallel DN$ and $NH_1 \parallel DM$, $MDNH_1$ is a parallelogram. Similarly, $NH_2 \parallel CM$ and $MH_2 \parallel CN$ imply $MCNH_2$ is a parallelogram. Let P be the midpoint of the segment $\overline{MN}$. Then $\sigma_P(D) = H_1$ and $\sigma_P(C) = H_2$, thus $CD \parallel H_1H_2$ and $\overline{CD} = \overline{H_1H_2}$.

From $CD \perp AB$ we deduce $A_{AH_1BH_2} = \frac{1}{2}\overline{AB} \cdot \overline{CD} = S$.

Problem 3. *Let a,b,c be positive real numbers such that $abc=1$. Prove that*

$$\left(a+\frac{1}{b}\right)^2+\left(b+\frac{1}{c}\right)^2+\left(c+\frac{1}{a}\right)^2\geq 3\left(a+b+c+1\right).$$

When does equality hold?

Solution 1. By using AM-GM ($x^2+y^2+z^2\geq xy+yz+zx$) we have

$$\begin{aligned}\left(a+\frac{1}{b}\right)^2+\left(b+\frac{1}{c}\right)^2+\left(c+\frac{1}{a}\right)^2 &\geq \left(a+\frac{1}{b}\right)\left(b+\frac{1}{c}\right)+\left(b+\frac{1}{c}\right)\left(c+\frac{1}{a}\right)+\left(c+\frac{1}{a}\right)\left(a+\frac{1}{b}\right)\\ &=\left(ab+1+\frac{a}{c}+a\right)+\left(bc+1+\frac{b}{a}+b\right)+\left(ca+1+\frac{c}{b}+c\right)\\ &=ab+bc+ca+\frac{a}{c}+\frac{c}{b}+\frac{b}{a}+3+a+b+c.\end{aligned}$$

Notice that by AM-GM we have $ab+\frac{b}{a}\geq 2b$, $bc+\frac{c}{b}\geq 2c$, and $ca+\frac{a}{c}\geq 2a$.

Thus ,

$$\left(a+\frac{1}{b}\right)^2+\left(b+\frac{1}{c}\right)^2+\left(c+\frac{1}{a}\right)^2\geq\left(ab+\frac{b}{a}\right)+\left(bc+\frac{c}{b}\right)+\left(ca+\frac{a}{c}\right)+3+a+b+c\geq 3(a+b+c+1).$$

The equality holds if and only if $a=b=c=1$.

Solution 2. From QM-AM we obtain

$$\sqrt{\frac{\left(a+\frac{1}{b}\right)^2+\left(b+\frac{1}{c}\right)^2+\left(c+\frac{1}{a}\right)^2}{3}}\geq\frac{a+\frac{1}{b}+b+\frac{1}{c}+c+\frac{1}{a}}{3}\Leftrightarrow$$

$$\left(a+\frac{1}{b}\right)^2+\left(b+\frac{1}{c}\right)^2+\left(c+\frac{1}{a}\right)^2\geq\frac{\left(a+\frac{1}{b}+b+\frac{1}{c}+c+\frac{1}{a}\right)^2}{3}\quad(1)$$

From AM-GM we have $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\geq 3\sqrt[3]{\frac{1}{abc}}=3$, and substituting in (1) we get

$$\left(a+\frac{1}{b}\right)^2+\left(b+\frac{1}{c}\right)^2+\left(c+\frac{1}{a}\right)^2\geq\frac{\left(a+\frac{1}{b}+b+\frac{1}{c}+c+\frac{1}{a}\right)^2}{3}\geq\frac{\left(a+b+c+3\right)^2}{3}=$$

$$=\frac{\left(a+b+c\right)\left(a+b+c\right)+6\left(a+b+c\right)+9}{3}\geq\frac{\left(a+b+c\right)3\sqrt[3]{abc}+6\left(a+b+c\right)+9}{3}=$$

$$=\frac{9\left(a+b+c\right)+9}{3}=3\left(a+b+c+1\right).$$

The equality holds if and only if $a=b=c=1$.

Solution 3.

By using $x^2+y^2+z^2\geq xy+yz+zx$

$$\left(a+\frac{1}{b}\right)^2+\left(b+\frac{1}{c}\right)^2+\left(c+\frac{1}{a}\right)^2=a^2+b^2+c^2+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{a^2}+\frac{2a}{b}+\frac{2b}{c}+\frac{2c}{a}\geq$$
$$\geq ab+ac+bc+\frac{1}{bc}+\frac{1}{ca}+\frac{1}{ab}+\frac{2a}{b}+\frac{2b}{c}+\frac{2c}{a}.$$

Clearly

$$\frac{1}{bc}+\frac{1}{ca}+\frac{1}{ab}=\frac{abc}{bc}+\frac{abc}{ca}+\frac{abc}{ab}=a+b+c,$$
$$ab+\frac{a}{b}+bc+\frac{b}{c}+ca+\frac{c}{a}\geq 2a+2b+2c,$$
$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq 3\sqrt[3]{\frac{a}{b}\cdot\frac{b}{c}\cdot\frac{c}{a}}=3.$$

Hence

$$\left(a+\frac{1}{b}\right)^2+\left(b+\frac{1}{c}\right)^2+\left(c+\frac{1}{a}\right)^2\geq\left(ab+\frac{a}{b}\right)+\left(ac+\frac{c}{a}\right)+\left(bc+\frac{b}{c}\right)+a+b+c+\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq$$
$$\geq 2a+2b+2c+a+b+c+3=3\left(a+b+c+1\right).$$

The equality holds if and only if $a=b=c=1$.

Solution 4. $a=\frac{x}{y}$, $b=\frac{y}{z}$, $c=\frac{z}{x}$

$$\left(\frac{x}{y}+\frac{z}{y}\right)^2+\left(\frac{y}{z}+\frac{x}{z}\right)^2+\left(\frac{z}{x}+\frac{y}{x}\right)^2\geq 3\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}+1\right)$$

$$(x+z)^2x^2z^2+(y+x)^2y^2x^2+(z+y)^2z^2y^2\geq 3xyz(x^2z+y^2x+z^2y+xyz)$$

$$x^4z^2+2x^3z^3+x^2z^4+x^2y^4+2x^3y^3+x^4y^2+y^2z^4+2y^3z^3+y^4z^2\geq 3x^3yz^2+3x^2y^3z+3xy^2z^3+3x^2y^2z^2$$

$1)x^3y^3+y^3z^3+z^3x^3\geq 3x^2y^2z^2$.

$$\left.\begin{array}{l}2)x^4z^2+z^4x^2+x^3y^3\geq 3x^3z^2y\\ 3)x^4y^2+y^4x^2+y^3z^3\geq 3y^3x^2z\\ 4)z^4y^2+y^4z^2+x^3z^3\geq 3z^3y^2x\end{array}\right\}$$

Equality holds when $x=y=z$, i.e., $a=b=c=1$.

Solution 5. $\sum_{cyc}(a+\frac{1}{b})^2\geq 3\sum_{cyc}a+3$

$$\Leftrightarrow 2\sum_{cyc}\frac{a}{b}+\sum_{cyc}\left(a^2+\frac{1}{a^2}-3a-1\right)\geq 0$$

$$2\sum_{cyc}\frac{a}{b}\geq 6\sqrt[3]{\frac{a}{b}\frac{b}{c}\frac{c}{a}}=6 \qquad (1)$$

$$\forall a > 0,\ a^2 + \frac{1}{a^2} - 3a \geq \frac{3}{a} - 4$$
$$\Leftrightarrow a^4 - 3a^3 + 4a^2 - 3a + 1 \geq 0$$
$$\Leftrightarrow \left(a-1\right)^2\left(a^2 - a + 1\right) \geq 0$$
$$\sum_{cyc}\left(a^2 + \frac{1}{a^2} - 3a - 1\right) \geq 3\sum_{cyc}\frac{1}{a} - 15 \geq 9\sqrt[3]{\frac{1}{abc}} - 15 = -6 \qquad (2)$$
Using (1) and (2) we obtain
$$2\sum_{cyc}\frac{a}{b} + \sum\left(a^2 + \frac{1}{a^2} - 3a - 1\right) \geq 6 - 6 = 0$$
Equality holds when $a = b = c = 1$.

Problem 4. For a positive integer n, two players A and B play the following game: Given a pile of s stones, the players take turn alternatively with A going first. On each turn the player is allowed to take either one stone, or a prime number of stones, or a multiple of n stones. The winner is the one who takes the last stone. Assuming both A and B play perfectly, for how many values of s the player A cannot win?

Solution. Denote by k the sought number and let $\{s_1, s_2, ..., s_k\}$ be the corresponding values for s. We call each s_i a losing number and every other nonnegative integer a winning numbers.

Clearly every multiple of n is a winning number.

Suppose there are two different losing numbers $s_i > s_j$, which are congruent modulo n. Then, on his first turn of play, player A may remove $s_i - s_j$ stones (since $n | s_i - s_j$), leaving a pile with s_j stones for B. This is in contradiction with both s_i and s_j being losing numbers.

Hence, there are at most $n-1$ losing numbers, i.e. $k \le n-1$.

Suppose there exists an integer $r \in \{1, 2, ..., n-1\}$, such that $mn + r$ is a winning number for every $m \in \mathbb{N}_0$. Let us denote by u the greatest losing number (if $k > 0$) or 0 (if $k = 0$), and let $s = LCM(2, 3, ..., u + n + 1)$. Note that all the numbers $s+2$, $s+3$, ..., $s + u + n + 1$ are composite. Let $m' \in \mathbb{N}_0$, be such that $s + u + 2 \le m'n + r \le s + u + n + 1$. In order for $m'n + r$ to be a winning number, there must exist an integer p, which is either one, or prime, or a positive multiple of n, such that $m'n + r - p$ is a losing number or 0, and hence lesser than or equal to u. Since $s + 2 \le m'n + r - u \le p \le m'n + r \le s + u + n + 1$, p must be a composite, hence p is a multiple of n (say $p = qn$). But then $m'n + r - p = (m' - q)n + r$ must be a winning number, according to our assumption. This contradicts our assumption that all numbers $mn + r$, $m \in \mathbb{N}_0$ are winning.

Hence, each nonzero residue class modulo n contains a loosing number.

There are exactly $n-1$ losing numbers .

Lemma: No pair (u,n) of positive integers satisfies the following property:

(*) In $\mathbb{N}$ exists an arithmetic progression $(a_t)_{t=1}^{\infty}$ with difference n such that each segment

$\left[a_i - u, a_i + u\right]$ contains a prime.

Proof of the lemma: Suppose such a pair (u,n) and a corresponding arithmetic progression $(\mathrm{a}_t)_{t=1}^{\infty}$ exist. In $\mathbb{N}$ exist arbitrarily long patches of consecutive composites. Take such a patch P of length $3un$. Then, at least one segment $\left[a_i - u, a_i + u\right]$ is fully contained in P, a contradiction.

Suppose such a nonzero residue class modulo n exists (hence $n > 1$). Let $u \in \mathbb{N}$ be greater than every loosing number. Consider the members of the supposed residue class which are greater than u. They form an arithmetic progression with the property (*) , a contradiction (by the lemma).